A Probability Model for Lifetime of Wireless Sensor Networks

Moslem Noori and Masoud Ardakani
Department of Electrical and Computer Engineering, University of Alberta, CANADA
Email: {moslem, ardakani}@ece.ualberta.ca

***Abstract*— Considering a wireless sensor network whose nodes are distributed randomly over a given area, a probability model for the network lifetime is provided. Using this model and assuming that packet generation follows a Poisson distribution, an analytical expression for the complementary cumulative density function (ccdf) of the lifetime is obtained. Using this ccdf, one can accurately find the probability that the network achieves a given lifetime. It is also shown that when the number of sensors, N, is large, with an error exponentially decaying with N, one can predict whether or not a certain lifetime can be achieved. The results of this work are obtained for both multi-hop and single-hop wireless sensor networks and are verified with computer simulation. The approaches of this paper are shown to be applicable to other packet generation models and the effect of the area shape is also investigated.**

I. Introduction

Wireless sensor networks (WSNs) are consisted of a set of cheap and usually battery-powered devices, called sensors. Sensor limited power usually necessitates a compromise between lifetime and other parameters such as the data rate or the quality of the received signal in the sink. It is usually impracticable to replace the sensors batteries after their operation period. Hence, estimating the network lifetime according to the initial energy in sensors is essential for network design. According to such lifetime estimation, one can choose the network parameters such as node density, data rate and initial energy of the sensors to achieve the desired lifetime.

Lifetime analysis has been studied in the literature based on different definitions such as the number of dead nodes in the network, network coverage and network connectivity [1]–[6]. Authors in [1] derive an upper bound on the network lifetime considering the spatial behavior of the data source. To achieve this goal, they first consider a simplified version where the data source is a specific point, and the source is connected to the sink with a straight line consisting of relaying sensors. They derive the optimum length of a hop and consequently the number of hops in the path to minimize the total energy consumed for the data delivery. Then, they remove the assumption of a source concentrated on a point and assume that the source is distributed over an area.

In [2], the results of [1] are extended to the networks whose nodes may perform different tasks of sensing, relaying and aggregating. The results of [1] are also extended to multiple-sink networks in [3].

Work reported in [4] studies the network lifetime for a cell based network. It is assumed that N nodes are deployed over a hypercube. For the aim of energy conserving, the area is divided to n hypercubes (cells). Using occupancy theory [7], the distribution of the minimum number of sensors within each cell is investigated when $N, n \to \infty$. Then, authors study the lifetime for the case when network remains almost surely connected. Using the number of sensors in each cell, the network lifetime is lower bounded based on the given lifetime of each sensor.

A lifetime study based on the area coverage is presented in [5]. It is assumed that the nodes have a circular sensing region and are distributed over a squared area. Using the stochastic geometry, theory of coverage process, and assuming the size of the area goes to infinity, an expression for the node density is derived to guarantee a k-coverage in the area. It is shown that using the proposed density, the network lifetime is upper bounded by kT where T is the given lifetime of each sensor. Although the upper bound is derived for an asymptotic situation when the area goes to infinity, it is shown through simulation that the derived bound is also reasonable for networks over a finite area.

Authors in [6] divide linear or circular networks to some bins where each bin contains a deterministically assigned number of nodes. The nodes within each bin, however, are deployed randomly. Also, the lifetime is defined as the time when a hole occurs in the routing scheme (i.e. death of a bin). Assuming a fixed transmission power for each packet and using the theory of stochastic processes, authors have found the probability distribution function (pdf) of the network lifetime. In addition, they propose a method to assign the number of nodes within each bin in order to maximize the network lifetime.

It is worthy to note that other studies in the literature are performed on the lifetime, e.g. [8]–[12]. However, the most related ones to this work are those that we discussed earlier.

In this paper, we find the probability of reaching a certain lifetime for randomly distributed networks based on the power dissipation model of the sensors. More specifically, unlike [4], [5], we do not assume that the lifetime of a sensor is given in order to find the network lifetime. Instead, we find the lifetime of a sensor (as a random variable) based on its power dissipation and packet generation model. Also, our analysis does not assume an infinite area and infinite number of sensors. In comparison to [6], we consider totally randomly deployed networks over more variant area shapes. In addition, both fixed and adjustable transmission power are studied in this work.

Also, the definition of lifetime in our work is more general and can include the case studied in [6] (to be discussed in Section IV).

Considering the randomness in packet generation and sensor deployment in the area, the lifetime of a network is a random variable. For a lifetime analysis of the network, it is needed to have a knowledge of the lifetime of each individual sensor. In this work, instead of assuming that the lifetime of each sensor is given beforehand, we first perform a lifetime analysis at the sensor level. To this end, we model the lifetime of a sensor as a random variable and find its distribution based on the traffic model and the power dissipation model in the sensor. Using this probabilistic model of a sensor lifetime and the distribution of the sensors over the area, the complementary cumulative distribution function (ccdf) of the lifetime of a single-hop network is derived. From this ccdf, the probability distribution function (pdf) of the lifetime is also obtained. The single-hop analysis will be the base of our further extensions.

In the proposed analysis, no asymptotic assumption is made on the number of nodes. Nevertheless, an asymptotic analysis is provided, which—with an error exponentially decaying with the number of sensors—predicts whether or not a desired lifetime can be achieved.

The above analysis is then extended to multi-hop networks. Since the lifetime of the multi-hop networks is dependent on the routing scheme, we study the lifetime ccdf under the *maximum-lifetime* [13] routing.

The methodologies of this work are applicable to more general scenarios, some of them are discussed in this paper. For example, we extend the results to different traffic models; to the case where different sensors may have different initial energy or traffic load; and to various area shapes.

The organization of this paper is as follows. In Section II we introduce the system model and provide the required definitions and assumptions. The lifetime analysis for single-hop networks is studied in Section III. Section IV discuses the lifetime analysis of multi-hop networks. Extensions to other scenarios are discussed in Section V and the accuracy of the proposed method is verified through simulations in Section VI. The paper is concluded in Section VII.

II. System Model

In this section, the components of the system model such as lifetime definition, energy consumption model and network traffic model are introduced.

A. Lifetime Definition

As mentioned, lifetime has a great significance in the design of WSNs. Conceptually, lifetime means the time duration that the network is operational and can perform its assigned task. Since there is no unique measure of the network failure, the definition of the lifetime is application-related.

In [14]–[16] lifetime is stated as the time when the first node dies. Usually the remaining sensors in the network can accomplish the network's assigned task. Therefore, another definition based on the ratio of dead nodes to the total number of nodes in the network is often used (e.g. [8], [17], [18]). Notice that this definition includes the definition of lifetime based on the death of the first node and therefore is more general. Other definitions based on the communication connectivity or the coverage of the area are also proposed for the lifetime [4], [5].

In this study, we consider the network lifetime based on the ratio of dead nodes to the total number of nodes, β. For multi-hop networks, where the nodes close to the sink have more traffic load than other nodes and die sooner, we will modify this definition.

B. Energy Consumption Model

The network lifetime is directly related to the sensors lifetime and in other words the energy dissipated in the sensor nodes. The consumed energy in sensors includes the energy required for sensing, receiving, transmitting and processing of data. The total consumed energy is usually dominated by the required energy for data transmission.

Two cases may be considered for the transmission mode of the nodes in the network. In the first case, nodes transmit with a fixed transmission power. This usually results in a fixed transmission range. In the second case, nodes use a mechanism to adjust their transmission power based on their distance to the next hop or the sink. Hence, the required energy for a packet transmission in sensor i can be modeled as [19]

$$\begin{aligned} e(d_i) &= l(e_t d_i^{\alpha} + e_o) \\ &= k d_i^{\alpha} + c \end{aligned} \tag{1}$$

where l represents the packet length in bits, d_i denotes the distance between sensor i and the next hop, α represents the path loss exponent, e_t shows the loss coefficient related to 1 bit transmission and e_o is the overhead energy due to the sensing, receiving and processing for the same amount of data. Also, $k = le_t$ and $c = le_o$ represent the loss coefficient and the overhead energy for a packet transmission respectively. The path loss exponent depends on the local terrain and is determined by empirical measurements. The typical value of α for WSNs is from 2 to 4 [18].

While this work is more focused on the transmission model (1), fixed transmission power is also discussed.

C. Traffic Model

The traffic model of the network depends on the network application and the behavior of sensed events. The data reporting process in WSNs is usually classified into three categories: event-driven, time-driven and query-driven [13]. In the time-driven case, sensors send their data periodically to the sink. Event-driven networks are used when it is desired to inform the data sink about the occurrence of an event. In query-driven networks, sink sends a request of data gathering when needed. In this paper, our main focus will be on the event-driven networks with Poisson model for packet generation.

Suppose that the events are independent (both temporally and spatially) and occur with equal probability over the area. In this case, Poisson distribution can be used effectively to

model the generation of data packets [6]. When the average rate of packet generation, λ, is known, the distribution of the number of data packets, M, generated by each node, from time 0 to T is

$$P(M = m) = \frac{e^{-\lambda T}(\lambda T)^m}{m!} \tag{2}$$

where m is a nonnegative integer number. Since the packet generation distribution obeys the Poisson model, the time duration between two consequent packet transmissions, t, has an exponential distribution with mean $\frac{1}{\lambda}$:

$$f_t(x) = \lambda e^{-x\lambda} u(x) \tag{3}$$

where $u(x)$ denotes the unit step function.

We will consider the Poisson model for sensor's traffic in this study. However, the proposed method can be extended to other traffic distributions and data gathering scenarios.

III. Lifetime Analysis in Single-hop Networks

In this section, we derive the pdf of the lifetime in single-hop WSNs. Assuming that nodes directly communicate with the sink, we first derive the ccdf of the lifetime. Then, the pdf of the lifetime is obtained by taking the derivative of the ccdf. The results are extended to the case of multi-hop networks in Section IV.

It is assumed here that all of the nodes have the same initial energy, same distribution over the area and the same packet generation model. Other cases like nonuniform energy distribution or different packet generation models are studied in Section V.

For the ease of presentation, the list of parameters is provided in Table I. As mentioned, the lifetime of a single-hop WSN is considered as the time when the ratio of dead nodes to the total number of nodes, N, passes a threshold, β.

N	Number of deployed nodes in the area
β	Threshold for the ratio of dead nodes to all nodes
α	Path loss exponent
k	Path loss coefficient
c	Overhead energy
E_i	Initial energy in sensors
τ	Lifetime threshold
t_i	Lifetime achieved by sensor i
L	Lifetime achieved by the network
λ	Average rate of packet generation
d_i	Distance of sensor i to the next hop

TABLE I

Parameters of the problem

We start the network lifetime analysis by considering the lifetime of one sensor. Defining p_i as

$$p_i = \frac{E_i}{e(d_i)} \tag{4}$$

for sensor i, it is clear that the maximum number of packets that can be transmitted by this sensor is equal to $\lfloor p_i \rfloor$.

Lemma 1: If a sensor node with initial energy E_i is randomly placed in the area $\mathcal{R}$, the probability of achieving a lifetime more than a threshold τ will be

$$P(t_i \geq \tau) = 1 - \frac{\gamma(\lfloor p_i \rfloor, \lambda\tau)}{\Gamma(\lfloor p_i \rfloor)} \tag{5}$$

where $\gamma(\cdot,\cdot)$ denotes the lower incomplete gamma function

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt \tag{6}$$

and $\Gamma(\cdot)$ represents the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{7}$$

Proof: The lifetime of sensor i, t_i, depends on the maximum number of packets that can be transmitted by the sensor to the sink. Since t_i is the sum of time durations between packet transmissions until the last packet is sent by the sensor, we have

$$t_i = \sum_{j=1}^{\lfloor p_i \rfloor} t_{ij} \tag{8}$$

where t_{ij} denotes the time duration between transmitting packets $j-1$ and j by sensor i, and t_{i1} is defined as the time when the first packet is transmitted. Since a Poisson model is assumed for data packet generation, t_{ij}'s obey an exponential distribution indicated in (3). On the other hand, it is known that the sum of independent identically distributed (i.i.d) exponential random variables has a gamma distribution [20]. It is worthy to note that since the node is deployed randomly in the area, the distance between the node and the sink and consequently p_i are a random variables. Hence, given p_i, the conditional pdf of t_i can be written as follows

$$f_{t_i|p_i}(x) = \lambda^{\lfloor p_i \rfloor} \frac{x^{\lfloor p_i \rfloor - 1} e^{-\lambda x}}{\Gamma(\lfloor p_i \rfloor)} \quad x \geq 0. \tag{9}$$

Now

$$\begin{aligned} P(t_i \geq \tau | p_i) &= 1 - \int_0^\tau \lambda^{\lfloor p_i \rfloor} \frac{x^{\lfloor p_i \rfloor - 1} e^{-\lambda x}}{\Gamma(\lfloor p_i \rfloor)}\, dx \\ &= 1 - \frac{\gamma(\lfloor p_i \rfloor, \lambda\tau)}{\Gamma(\lfloor p_i \rfloor)}. \end{aligned} \tag{10}$$

■

Proposition 1: Since the fractional part of p_i is usually much smaller than the integer part, $\lfloor p_i \rfloor \simeq p_i$ and hence (10) can be rewritten as

$$P(t_i \geq \tau | p_i) = 1 - \frac{\gamma(p_i, \lambda\tau)}{\Gamma(p_i)} \tag{11}$$

For simplicity, we use (11) to analyze the network lifetime in the sequel. □

Corollary 1: In the case of the fixed transmission range, r, each node lives more than the threshold with probability

$$P(t_i \geq \tau) = 1 - \frac{\gamma(p_f, \lambda\tau)}{\Gamma(p_f)} \tag{12}$$

where

$$p_f = \frac{E_i}{kr^\alpha + c}. \tag{13}$$

Proof: In this case, all of the p_i's have a deterministic value equal to p_f. Therefore, the value of $P(t_i \geq \tau)$ in (11) is unconditional and the proof is completed by replacing p_i by p_f in (11). □

One can take another approach and approximate the value of $P(t_i \geq \tau)$ to find a simpler form of (11).

Proposition 2: Since t_i in (8) is the sum of i.i.d. random variables, central limit theorem (CLT) [20] indicates that its pdf tends to Gaussian distribution with mean $\lfloor p_i \rfloor \lambda^{-1}$ and variance $\lfloor p_i \rfloor \lambda^{-2}$. Considering $\lfloor p_i \rfloor \approx p_i$, we have

$$P(t_i \geq \tau | p_i) = Q\left(\frac{\tau - p_i\lambda^{-1}}{\sqrt{p_i}\lambda^{-1}}\right). \tag{14}$$

where $Q(\cdot)$ is the ccdf of the normal distribution. □

To study the lifetime of the network, we consider the lifetime of all the nodes in the network which necessitates the knowledge of p_i for all of the nodes in the network. When a node is deployed randomly over an area, p_i is a random variable with pdf $f_{p_i}(x)$. In a random network deployment, p_i's are usually i.i.d. random variables and consequently have the same distribution, $f_p(x)$. This distribution depends on the shape of the area, energy dissipation model and the pdf of node distribution over the area. In the Appendix, $f_p(x)$ is derived for some common area shapes assuming a uniform distribution for the node deployment.

Theorem 1: Assuming N equal-energy nodes are distributed independently over the area $\mathcal{R}$, the probability that the network achieves a lifetime more than a given threshold, τ, is equal to

$$P(L \geq \tau) = Q\left(\sqrt{N}\frac{1-\beta-\mu}{\sigma}\right) \tag{15}$$

where

$$\mu = \int_{\mathcal{R}} \left(1 - \frac{\gamma(x, \lambda\tau)}{\Gamma(x)}\right) f_p(x)\, dx \tag{16}$$

$$\sigma = \sqrt{\mu - \mu^2}. \tag{17}$$

Proof: To find the number of nodes that live more than the lifetime threshold, we define a Bernoulli random variable l_i indicating the success of achieving the lifetime threshold by sensor i:

$$l_i = \begin{cases} 1 & \text{With probability equal to } s_i, \\ 0 & \text{With probability equal to } 1 - s_i. \end{cases} \tag{18}$$

The success probability of l_i, given p_i, is equal to

$$s_i = P(t_i \geq \tau | p_i) = 1 - \frac{\gamma(p_i, \lambda\tau)}{\Gamma(p_i)} \tag{19}$$

which was derived in Lemma 1. The number of live nodes after time τ can be found by defining a new random variable, w, that denotes the number of successes in the Bernoulli trials shown by l_i's

$$w = \sum_{i=1}^{N} l_i. \tag{20}$$

Since nodes packet generations are independent and p_i's are i.i.d., s_i's and consequently l_i's are also i.i.d random variables. In this case, w has a binomial distribution [21]. Also, when the number of trials is large enough, one can approximate the binomial distribution with a Gaussian distribution. Since the number of nodes are usually large enough, CLT can be applied on (20). Hence

$$f_w(x) = \frac{1}{\sqrt{2\pi}\sigma_w} \exp - \frac{(x-\mu_w)^2}{2\sigma_w^2} \tag{21}$$

where μ_w is the mean and σ_w^2 denotes the variance of w. From (20), it is clear that

$$\mu_w = \sum_{i=1}^{N} \mu_{l_i} \tag{22}$$

where μ_{l_i} is the mean of l_i. Since l_i's are independent random variables

$$\sigma_w^2 = \sum_{i=1}^{N} \sigma_{l_i}^2 \tag{23}$$

where $\sigma_{l_i}^2$ is the variance of l_i. To find the values of μ_w and σ_w, we need to have the unconditional mean and variance of l_i's using the conditional values. Since l_i's are Bernoulli random variables

$$\mu_{l_i|p_i} = s_i, \quad \sigma_{l_i|p_i}^2 = s_i - s_i^2. \tag{24}$$

On the other hand, for two random variables x and z, the unconditional mean and variance of x can be found using the conditional mean and variance as follows [20]

$$\mu_x = E[\mu_{x|z}] \tag{25}$$

$$\sigma_x^2 = E[\sigma_{x|z}^2] + \mathrm{Var}[\mu_{x|z}] \tag{26}$$

where $E[\cdot]$ is the expected value and $\mathrm{Var}[\cdot]$ denotes the variance of the random variable. Using (19), (24), (25) and (26), it can be shown that

$$\mu_{l_i} = E[s_i] = \int_{\mathcal{R}} \left(1 - \frac{\gamma(x, \lambda\tau)}{\Gamma(x)}\right) f_{p_i}(x)\, dx \tag{27}$$

$$\sigma_{l_i}^2 = E[s_i - s_i^2] + \mathrm{Var}[s_i] = E[s_i] - E^2[s_i] = \mu_{l_i} - \mu_{l_i}^2. \tag{28}$$

Since p_i's are i.i.d random variables with pdf $f_p(x)$, we have

$$\mu_{l_i} = \mu = \int_{\mathcal{R}} \left(1 - \frac{\gamma(x, \lambda\tau)}{\Gamma(x)}\right) f_p(x)\, dx \quad \forall i \tag{29}$$

$$\sigma_{l_i} = \sigma = \sqrt{\mu - \mu^2} \quad \forall i. \tag{30}$$

Then, using (22) and (23)

$$\mu_w = N\mu, \quad \sigma_w^2 = N\sigma^2 \tag{31}$$

To derive the probability of achieving the lifetime threshold by the network, we just need to know the probability of achieving the lifetime by at least $(1-\beta)N$ nodes. Hence

$$F_L^c(\tau) = P(L \geq \tau) = P(w \geq (1-\beta)N) = Q\left(\sqrt{N}\frac{1-\beta-\mu}{\sigma}\right) \tag{32}$$

where $F_L^c(\tau)$ represents the ccdf of the network lifetime. ■

Proposition 3: Using Proposition 2, μ can also be calculated as

$$\mu = \int_{\mathcal{R}} Q\left(\frac{T_{\rm thr} - x\lambda^{-1}}{\sqrt{x}\lambda^{-1}}\right) f_p(x)\, dx. \tag{33}$$

□

Corollary 2: Assuming a network with parameters given in Theorem 1, the probability distribution function of the network lifetime is

$$f_L(\tau) = \frac{\lambda\sqrt{N}}{2\sqrt{2\pi}} \frac{1-\mu-\beta(1-2\mu)}{(\mu-\mu^2)^{\frac{3}{2}}} c(\tau) e^{-(\lambda\tau + \frac{N(1-\beta-\mu)^2}{2(\mu-\mu^2)})} \quad 0 \le \tau \le \infty \tag{34}$$

where

$$c(\tau) = \int_{\mathcal{R}} \frac{f_p(x)}{\Gamma(x)} (\lambda\tau)^{x-1}\, dx. \tag{35}$$

Proof: The ccdf of the network lifetime was derived in the previous theorem. Then we have

$$f_L(\tau) = -\frac{d(F_L^c(\tau))}{d\tau} = -\frac{d(F_L^c(\mu))}{d\mu}\frac{d\mu}{d\tau} \tag{36}$$

which results in (34). ■

IV. Lifetime Analysis in Multi-hop Networks

In multi-hop networks, the network lifetime depends on the way that the routing scheme distributes the traffic load among the sensor nodes. The minimum cost routing (minimum required energy or minimum number of hops) is conventionally used in wireless networks. However, this routing scheme cannot guarantee the maximum lifetime in the network [13]. On the other hand, maximum lifetime routing attempts to prolong the network lifetime by proper traffic distribution among the nodes. This scheme may not have the minimum overall consumed energy. Since we mainly focus on the lifetime analysis, we just consider the maximum lifetime routing. Nevertheless, the proposed approach can be used for other routing schemes knowing how the traffic is distributed among the nodes.

In multi-hop networks, the whole network traffic passes through the nodes in the vicinity of the sink, hence, death of these nodes can have a significant effect on the network performance. Therefore, we need to modify our previous definition of the lifetime.

Assume that $\mathcal{H}$ shows the set of nodes that are in the vicinity of the sink and directly communicate with it. Since all other nodes communicate to the sink through these nodes, they will be out of energy sooner than the other ones. So, we define the lifetime based on the ratio of dead nodes within $\mathcal{H}$ to $|\mathcal{H}|$ where $|\cdot|$ denotes the cardinality of the set. It is also assumed that the sensors are distributed over a circle with radius R and the data sink is positioned at the center of the area. The lifetime of the network over other area shapes will be discussed later.

We assume that the sensors perform transmission with a fixed power which results in a fixed transmission range r. Based on the maximum transmission radius of the sensors and the area radius, the area can be divided to a number of rings (Figure 1). The sensors within a ring send their data to the sensors within the neighboring inner ring. The number of rings, n, within the area can be simply found as

$$n = \left\lceil \frac{R}{r} \right\rceil \tag{37}$$

where $\lceil\cdot\rceil$ denotes the integer ceiling. For simplicity, it is assumed that R is an integer multiple of r. This assumption allows us to focus on methodologies and can be removed if necessary. In addition, since each ring carries the traffic of all outer rings, the average traffic carried by the sensors within each ring is different and depends on the distance of the ring to the sink. To study the network lifetime, we consider the case when the routing scheme distributes the network traffic equally between the nodes within each ring. This scheme prevents the nodes from being exhausted quickly and prolongs the lifetime.

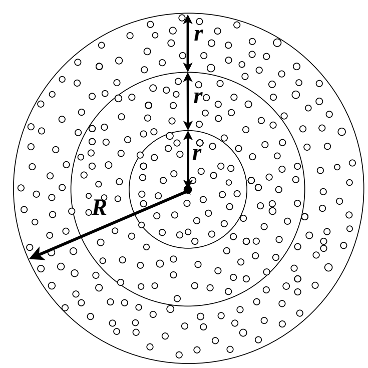

Fig. 1. Rings within a multi-hop network

Based on the assumed routing scheme, the average rate of the packet transmission by each node within ring i is equal to

$$\lambda_i = \lambda\frac{N - \sum_{j=1}^{i-1} N_j}{N_i} \quad \forall i = 1, 2, \ldots, n \tag{38}$$

where N_i denotes the number of sensors within ring i. Since nodes are assumed to be deployed randomly in the area, N_i is a binomial random variable. If one assumes a uniform deployment for the nodes, N_i will have a binomial distribution with mean Nq_i where

$$q_i = \frac{r^2(2i-1)}{R^2} \tag{39}$$

represents the probability of positioning a sensor in the ring i. Therefore, the time duration between two consequent transmissions, t, by a node in the ring i obeys an exponential distribution as follows

$$f_{t|N_i}(x) = \lambda_i e^{-x\lambda_i} u(x). \tag{40}$$

Since the lifetime is mainly effected by the nodes within the first tier, we just consider the probability of achieving the lifetime threshold by the first ring. Nevertheless, the probability of achieving τ by other rings can also be investigated using (40). As discussed earlier, probability of achieving a lifetime

threshold depends on the number of nodes within the area. Hence, using Theorem 1 and Corollary 1, one can find the conditional probability of achieving τ by the first ring

$$P(L \geq \tau | N_1) = Q\left(\sqrt{N_1}\frac{1-\beta-\mu}{\sigma}\right) \tag{41}$$

where

$$\mu = 1 - \frac{\gamma(p_f, \lambda_1 \tau)}{\Gamma(p_f)} \tag{42}$$

$$\sigma = \sqrt{\mu - \mu^2}. \tag{43}$$

Therefore, by removing the condition on N_1 in (41), we have

$$P(L \geq \tau) = \sum_{j=0}^{N} P(L \geq \tau | N_1) P(N_1 = j) \quad n_1 = 1, 2, \ldots, N \tag{44}$$

where

$$P(N_1 = n_1) = \left(\begin{array}{c} N \\ n_1 \end{array}\right) q_1^{n_1} (1-q_1)^{N-n_1}. \tag{45}$$

The given discussion is not restricted to circular areas and can also be applied to other area shapes. To study the lifetime of the network in other area shapes, we just need to recalculate the value of q_1 as follows

$$q_1 = \frac{\pi r^2}{S} \tag{46}$$

where S is the size of area. Then, ccdf of the lifetime is derived by putting this value of q_1 into (44).

V. Some Notes

In Section III, we considered the finite number of nodes in the area. We will study the asymptotic analysis in this section. Also, we earlier studied the case when all of the sensors have the same features such as traffic model, initial energy and deployment. In addition, the packet generation model was supposed to be Poisson. Here, we provide some discussions on the results in Section III and generalize them for more cases.

A. Asymptotic Analysis

Since the lifetime ccdf in (15) depends on the number of nodes distributed over the area, we can study the effect of the node density on the probability of achieving the lifetime threshold.

Corollary 3: The probability of achieving a lifetime threshold approaches 0 or 1 by increasing the number of nodes.

Proof: For large N, two cases can happen depending on the sign of $a = 1 - \beta - \mu$. Since Q-function is a decreasing function, when $a > 0$, increasing N causes the probability of hitting the lifetime threshold to tend $Q(\infty) = 0$. In other words, almost surely the given lifetime threshold cannot be achieved. Now, considering that [22]

$$\frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right) e^{-\frac{x^2}{2}} < Q(x) < \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} \quad \forall x \geq 0 \tag{47}$$

the rate of the probability decay is proportional to e^{-N}. In a similar manner, the probability approaches $Q(-\infty) = 1$ when $a < 0$. That is, the network almost surely achieves the lifetime threshold. The error in this prediction also decays exponentially with N. ■

An interesting case occurs when one considers the lifetime of the network based on the death of the first node. In this case $\beta = \frac{1}{N}$, which approaches 0 when N increases. Hence, according to the Corollary 3, it is necessary to consider just the sign of $1 - \mu$ in order to predict the asymptotic behavior of the network lifetime (i.e. when $N \to \infty$). Assuming $\tau > 0$, we have

$$\mu = \int_{\mathcal{R}} \left(1 - \frac{\gamma(x, \lambda\tau)}{\Gamma(x)}\right) f_p(x)\, dx < \int_{\mathcal{R}} f_p(x)\, dx = 1 \tag{48}$$

and consequently $1 - \mu > 0$. Therefore, under this stringent definition of the lifetime, the probability of achieving the lifetime τ approaches 0 as N increases.

B. Different Traffic Models

In Section III, we considered the case when all of the sensors have the same Poisson model for the packet generation. Here, we consider two other cases: 1) The average rate of packet generation changes with the position of the sensor, 2) packet generation obeys another model rather than Poisson. It is worthy to note that the assumed model is similar for all of the sensors.

If the average rate of packet generation, λ, varies with the position of the sensor (e.g. due to the spatial correlation of data or data aggregation and compression), we have the mean and variance of l_i conditioned on both p and λ. To derive the unconditional mean and variance of l_i, we need to calculate

$$\mu_{l_i} = \mu = \int \int_{\mathcal{R}} \left(1 - \frac{\gamma(x, \lambda y)}{\Gamma(x)}\right) f_{p,\lambda}(x, y)\, dxdy \tag{49}$$

where $f_{p,\lambda}(x, y)$ denotes the joint pdf of p and λ. Other parts of the analysis will remain unchanged.

Also, the proof given for Theorem 1 can be applied to the cases when the traffic model obeys another pattern rather than Poisson model. Assume that the pdf of the time duration between two packet transmissions follows a model with mean μ_t and variance σ_t^2. Using CLT, t_i can be accurately approximated by a Gaussian distribution with mean $p_i \mu_t$ and variance $p_i \sigma_t^2$. The remaining part of the proof is unchanged.

The proposed analysis can also be extended to time-driven networks. In this case, the time duration between two consequent transmissions is fixed and is equal to T. Hence

$$t_i = \lfloor p_i \rfloor T. \tag{50}$$

The unconditional values of μ and σ is found by integration over p_i. Then, the result given in Theorem 1 can be applied.

C. Nonuniform Energy Distribution

Assume that the energy is distributed over the network in a nonuniform way. As a consequence, s_i's in (19) are not identically distributed. This may also arise when the sensors generate packets with different rates (i.e. nonidentical Poisson distributions). In this situation, w does not have any standard

distribution, however, we can still use CLT to approximate the pdf of w with a Gaussian distribution. To this end, we will give a brief discussion on the probability of achieving the lifetime threshold by the network.

Lemma 2: Assume that z_i's ($1 \leq i \leq m$) are m independent random variables such that

$$\sum_{i=1}^{m} \mu_{z_i} = m\overline{\mu}. \tag{51}$$

where μ_{z_i} denotes the mean of z_i. Also, X_i's ($1 \leq i \leq m$) are m Bernoulli trials such that

$$P(X_i = 1) = z_i \quad \forall i. \tag{52}$$

Now, if X denotes the sum of X_i's, the variance of X is maximum when

$$\mu_{z_i} = \overline{\mu} \quad \forall i = 1, \ldots, m \tag{53}$$

(see [21] for the proof).

Corollary 4: For nonidentical distributed s_i's such that

$$\mu_w = \sum_{i=1}^{N} \mu_{l_i} = \sum_{i=1}^{N} E[s_i] = N\mu \tag{54}$$

(15) is an upper bound for the probability of achieving the lifetime when $1 - \beta - \mu > 0$, otherwise it is a lower bound.

Proof: Since we assumed the identical distribution in the proof of Theorem 1, Lemma 2 indicates that σ_w in (31) is the maximum possible variance of w. The proof is completed considering the decreasing property of the Q-function. ■

VI. EXPERIMENTAL RESULTS

In this section, we investigate the accuracy of the proposed analysis through some experiments. We first study the probability of achieving a lifetime threshold in single-hop networks. To this end, the simulations are performed over different area shapes with the same area size to investigate the effect of the area shape. In addition, the effect of the node density is studied. Moreover, simulations are performed to study the network lifetime in multi-hop networks. Through these simulations, it will be shown that how the transmission range and consequently the number of hops effect the lifetime of the network.

A. Single-hop Networks

The parameters of model (1) depend on the data rate, antenna height, antenna gain, etc. Typical values of e_t and e_o are given in [23]. For $\alpha = 4$, which we use in our simulations, the values of e_t and e_o are respectively 0.0013 pJ/bit/m^4 and 50 nJ/bit for a 1Mbps data stream. Here, It is assumed that the packets have 1000 bits length, hence, k =1.3 pJ/m^4 and c =50 μJ in (1).

Network has 500 nodes that are deployed uniformly and sink is positioned at the center of the area. Also, the packet generation model obeys the Poisson distribution and each sensor sends its packets directly to the sink. All of the sensors have the same initial energy equal to 11 mJ. Assuming that sensors send packet with the average rate of 1 packet/hour, the probability of achieving the lifetime of 100 hours is studied through simulation.

To investigate the effect of the area shape on the lifetime, the simulations are carried out over areas with the same size equal to $100\pi\text{m}^2$ but with different shapes. To decrease the final result variance and reach the proper confidence interval, the simulation is run 10000 times over each area and the results are averaged.

Figure 2 depicts the probability of achieving the lifetime threshold vs. the ratio of dead nodes over circular, hexagonal, squared and triangular areas. As it can be seen, the probability of achieving the lifetime threshold in circular, hexagonal and squared areas are very close. Since in a triangle, the distance of the sensors to the sink is more non-uniform and it has the largest circumcircle compared to other area shapes, triangle has a smaller probability to achieve the lifetime threshold.

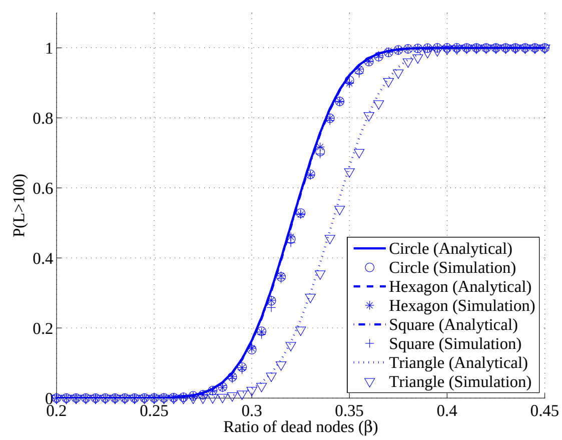

Fig. 2. Probability of achieving the lifetime threshold vs. the ratio of dead nodes for single-hop networks deployed over different area shapes

As discussed through the paper, depending on the value of $1 - \mu - \beta$ and by increasing the number of nodes, it can be almost surely determined whether the network achieves a lifetime threshold or not. The effect of the node density on achieving the lifetime threshold is shown in Figure 3. The lifetime of the network is considered as the moment when 0.3 of the nodes in the network die. In the first case, $E_i = 11$ mJ which results in $1 - \beta - \mu > 0$. Hence, as discussed in Section V, the desired probability decreases by increasing N which is verified by the simulation. In the second case, the initial energy is equal to 11.6 mJ which causes $1 - \beta - \mu < 0$. As shown in Figure 3, the probability of achieving the desired lifetime is an increasing function of N.

B. Multi-hop Networks

To study the network lifetime in a multi-hop network, it is assumed that 500 nodes are deployed uniformly over a circle with radius 100 m. All of the nodes have the same initial energy equal to $E_i = 100$ mJ. The parameters in (1) are kept the same as the previous part. A greedy routing algorithm is used to balance the network traffic such that data packets

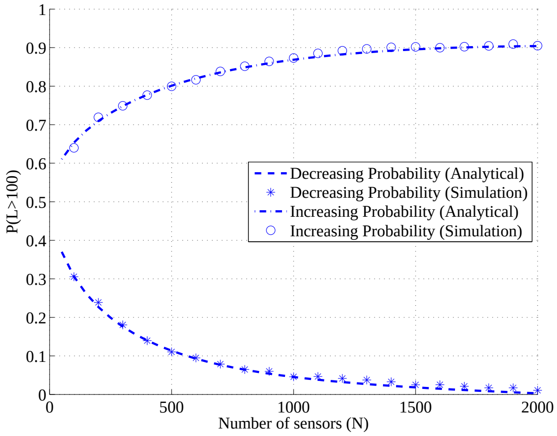

Fig. 3. Probability of achieving the lifetime threshold vs. the number of sensors in a single-hop network

are identically distributed between the nodes in the first ring of the network, $\mathcal{H}$. Considering this fact that all of the nodes use a constant transmission power and the traffic is distributed identically between the first-ring nodes, all of the nodes within $\mathcal{H}$ have approximately similar lifetime. As a consequence, they die in time moments very close to each other. Therefore, we can say that the desired probability is not significantly effected by the value of β (Figure 4).

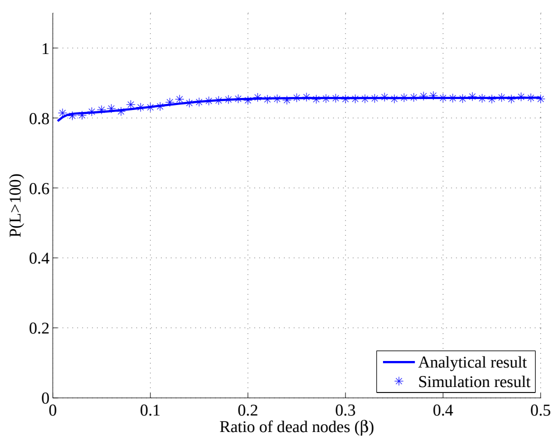

Fig. 4. Probability of achieving the lifetime threshold vs. the ratio of dead nodes in a multi-hop network

It is interesting to study the effect of the transmission range and consequently the number of hops on the lifetime. Figure 5 depicts the probability of reaching the lifetime threshold vs. the transmission range. The lifetime is considered as the moment when 0.3 nodes within $\mathcal{H}$ are dead. By decreasing r, number of nodes within $\mathcal{H}$ decreases, hence, they carry more packets and will die earlier. Therefore, it is expected that the desired probability decreases by reducing r. Indeed, while the nodes far from the sink still have enough energy to send packets, the nodes within $\mathcal{H}$ cease. To overcome this drawback, nonuniform energy distribution can be applied [24]. Also, the fixed transmission power causes the nodes within $\mathcal{H}$ to die sooner compared to the case where nodes adjust their transmission power.

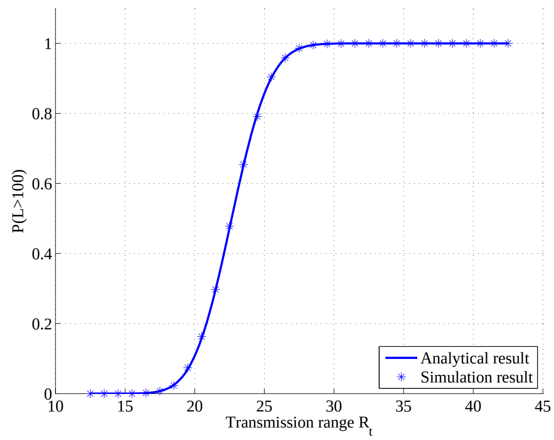

Fig. 5. Probability of achieving the lifetime threshold vs. the transmission range in a multi-hop network

VII. Conclusions

In this paper, we considered the problem of finding the probability of achieving a lifetime threshold by the network which is equivalent to finding the ccdf of the network lifetime. Using the power consumption model of (1), the ccdf of the lifetime was derived for the single-hop networks. To this end, it was assumed that all of the nodes have identical packet generation model, initial energy and random deployment in the area. The methodology was also extended to the case when these conditions may not be satisfied. Then, the problem was studied for the multi-hop case. In addition, the asymptotic relation between the number of nodes and the lifetime was investigated. Through some simulations, the accuracy of our analysis was investigated for the networks deployed over different area shapes. Using the proposed method, one can design both node and network parameters (e.g. node density, data rate, initial energy) according to the desired lifetime.

Appendix

The pdf of the network lifetime depends on the distribution of the maximum possible number of packet transmissions by each node, p. In this appendix, we find the pdf of p over some common area shapes. The pdf of p over a circle area is required for finding the lifetime pdf of multi-hop networks in Section IV. Also, this pdf over regular polygons is useful for studying the lifetime of a network composed of clusters tiling the area.

A. Network Deployed Over a Circle

Assume that the nodes are deployed uniformly over a circle with radius R. Also, assume that the sink is located at the center of the circle. Since the nodes are deployed uniformly over the area, the pdf of the distance between the nodes and sink, d, is

$$f_d(x) = \begin{cases} \frac{2x}{R^2} & 0 < x \leq R \\ 0 & \text{Otherwise} \end{cases}. \quad (55)$$

Now, using the energy consumption model (1), we have the following expression for the pdf of p

$$f_p(x) = \begin{cases} \frac{2E_i}{R^2 k\alpha x^2}\left[\frac{E_i - cx}{kx}\right]^{\frac{2-\alpha}{\alpha}} & \frac{E_i}{kR^\alpha + c} \le x < \frac{E_i}{c} \\ 0 & \text{Otherwise} \end{cases}. \quad (56)$$

B. Network Deployed Over a Regular Polygon

Suppose that the sensors are deployed over a regular polygon having n equal sides with length a. Again, we assume that the sink is placed at the center of the area. In this case, we have

$$f_d(x) = \begin{cases} \frac{2\pi x}{S} & 0 < x \le r_i \\ \frac{2\pi x - 2nx\cos^{-1}\frac{r}{x}}{S} & r_i < x \le R_c \\ 0 & \text{Otherwise} \end{cases} \quad (57)$$

where

$$r_i = \frac{a}{2}\cot\frac{\pi}{n} \quad (58)$$

is the radius of the inscribed circle of the polygon,

$$R_c = \frac{a}{2\sin\frac{\pi}{n}} \quad (59)$$

represents the radius of the circumcircle of the polygon and

$$S = \frac{n}{4}a^2\cot\frac{\pi}{n} \quad (60)$$

denotes the polygon area. Now, using the relation between d and p, we have

$$f_p(x) = \frac{2E_i}{k\alpha S x^2}\left[\frac{E_i - cx}{kx}\right]^{\frac{2-\alpha}{\alpha}}\left[\pi - n\cos^{-1}\frac{r}{\left(\frac{E_i - cx}{kx}\right)^{\frac{1}{\alpha}}}\right] \quad (61)$$

when

$$\frac{E_i}{kR_c^\alpha + c} \le x < \frac{E_i}{kr_i^\alpha + c}$$

and

$$f_p(x) = \frac{2\pi E_i}{k\alpha S x^2}\left[\frac{E_i - cx}{kx}\right]^{\frac{2-\alpha}{\alpha}} \quad (62)$$

when

$$\frac{E_i}{kr_i^\alpha + c} \le x < \frac{E_i}{c}$$

and $f_p(x) = 0$ elsewhere.

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